# Divide-and-Conquer Method: Problem Set

## 1 Introduction

**Divide-and-conquer**  In the divide-and-conquer method, we divide a problem into subproblems (of constant fraction size), solve each subproblem recursively, and combine the solutions to the subproblems to arrive at the solution to the problem.

To be efficient, it is important to balance the sizes of the subproblems.

**Searching**  Problems are usually stated in the form of searching for an object in a collection. In these situations, it may be useful to explicitly identify the possible scenarios. A typical algorithm for a search problem consists of a series of steps where in each step we perform some computation and ask a question to narrow down the possibilities. Each possible answer to the question reduces the problem to a subproblem where we have fewer possibilities for the object we are seeking. Moreover, the set of possible scenarios is partitioned along the possible answers to the question. For efficiency (that is, for minimizing the number of questions in the worst-case), it is important to design the algorithm so that the number of possibilities in each case is as equal as it can be to the number of possibilities in other cases.

# 2 Homework

**Problem 1: A Fake among 33 Coins**

Solve the following problems.

- There are $n = 33$ identical-looking coins; one of these coins is counterfeit and is known to be lighter than the genuine coins. Describe your algorithm for determining the fake coin. What is the minimum number of weighings needed to identify the fake coin with a two-pan balance scale without weights?

- We now consider the generalization of the previous problem for integers $n \geq 1$. There are $n$ identical-looking coins; one of these coins is counterfeit and is known to be lighter than the genuine coins. Describe your algorithm to identify the counterfeit. How many comparisons (as a function of $n$) does your algorithm require in the worst case?

**Problem 2: Coin Removal**

There is a line of $n$ coins on the table; some of them are heads up and the rest are tails up, in no particular order. The object of the puzzle is to remove all the coins by a sequence of moves. On each move, one can remove any head-up coin, after which its neighboring coin or coins, if any, must be turned over. Coins are considered "neighbors" if they are next to each other in the original line; if there is a gap between coins after some moves, the coins are no longer considered neighbors.

Determine the property of the starting line that is necessary and sufficient for the puzzle to have a solution. For those lines that can be removed by the puzzle's rules, design an algorithm for doing so.

# 3   Solved Problems

## Problem 3: Tromino Puzzle

Cover a $2^n \times 2^n$ ($n \geq 1$) board missing one square with right tromnioes, which are L-shaped tiles formed by three adjacent squares. The missing square can be any of the board squares. Trominoes should cover all the squares except the missing ones with no overlaps.

A right tromino can also be viewed as a $2 \times 2$ board with exactly one missing square.

## Solution: Tromino Puzzle

We will present an algorithm that covers a $2^n \times 2^n$ board with a missing square with $L$-shaped trominoes for all $n \geq 1$.

If $n = 1$, we have a $2 \times 2$ square with a missing square, which can be covered with one tromino.

If $n \geq 2$, we divide the board into four smaller boards, each of size $2^{n-1} \times 2^{n-1}$. However, of the four smaller boards, three of them do not miss any squares. We use an appropriately oriented tromino to cover the three corner squares (which meet at the center of the board) of the three smaller boards (the ones that do not miss any squares). We now have four smaller boards, each with exactly one missing square. We cover them recursively with $L$-shaped trominoes.

## Problem 4: A Fake among Eight Coins

There are eight identical-looking coins; one of these coins is counterfeit and is known to be lighter than the genuine coins. What is the minimum number of weighings needed to identify the fake coin with a two-pan balance scale without weights?

## Solution: A Fake among Eight Coins

Divide the coins into three groups of 3, 3, and 2 coins each and call them $L$, $R$, and $E$ respectively. Use the two-pan balance to compare the weights of the groups $L$ and $R$. If they weigh the same, the fake coin can neither be in $L$ nor in $R$, so it must be in $E$. Take the two coins in $E$ and compare them. The lighter coin (between the two coins in $E$) must be the fake coin.

If the weights of $L$ and $R$ are different, then the fake coin must be in the group that is lighter. Take any 2 coins from this group and weigh them. If they weigh the same then the remaining coin in the group must be fake. If one of them is lighter, the lighter coin must be the fake coin.

## Problem 5: Twelve Coins

There are 12 coins identical in appearance; either all are genuine or exactly one of them is fake. It is unknown whether the fake coin is lighter or heavier than the genuine one. You have a two-pan balance scale without weights. The problem is to find whether all the coins are genuine and, if not, to find the fake coin and establish whether it is lighter or heavier than the genuine ones. Design an algorithm to solve the problem in the minimum number of weighings.
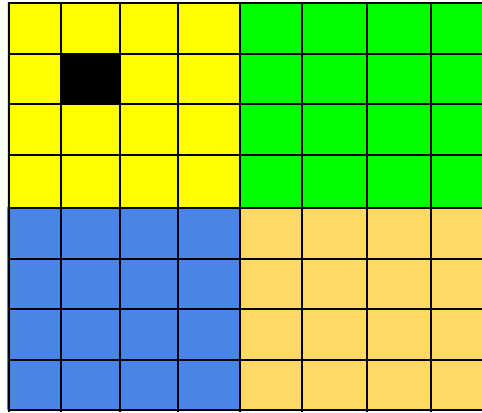
## Solution: Twelve Coins

Let $c_1, c_2, \ldots, c_{12}$ denote the coins. Let $L = \{c_1, c_2, c_3, c_4\}$, $R = \{c_5, c_6, c_7, c_8\}$, and $E = \{c_9, c_{10}, c_{11}, c_{12}\}$. Compare $L$ and $R$. We discuss each of the three cases below.
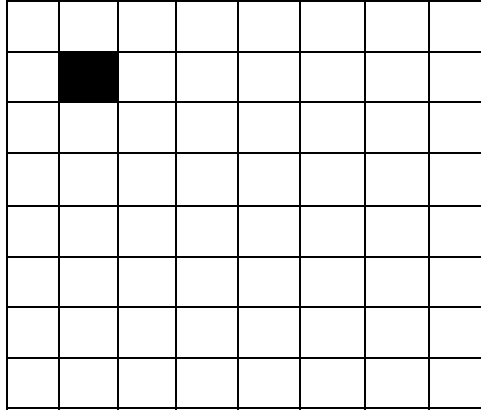
**8 x 8 Board with one missing square**
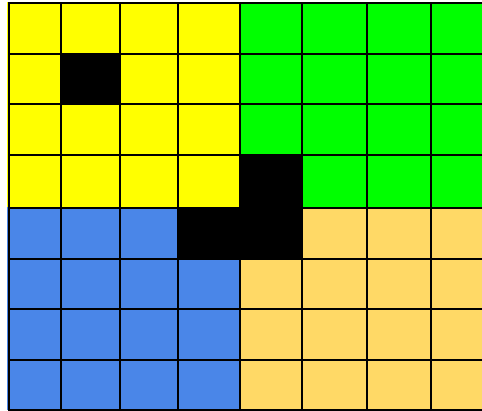
Figure 1: Tromino Puzzle

**8 x 8 Board with one missing square**

**Division into 4 equal quadrants (4 x 4 boards)**

**The subproblems of covering the quadrants are not exactly the same type as the original problem**
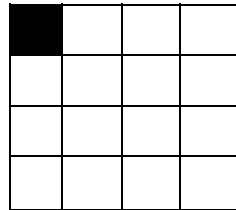
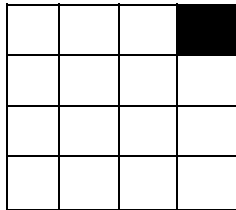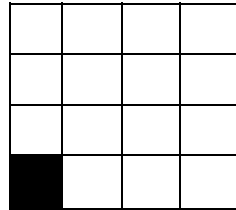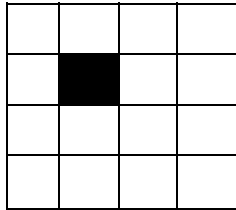Figure 2: Division into four subproblems- I

**8 x 8 Board with one missing square**

**Division into 4 equal smaller size boards (4 x 4 boards)**

**The subproblems of covering the smaller boards are not exactly the same type as the original problem since three of them do not have a missing square**

**Use a tromino to cover the three corner squares of the three smaller boards which do not have a missing square. The three corner squares meet at the center of the board.**

Figure 3: Division into four subproblems- II

**4 subproblems: 4 4x4 boards each with one missing square**

Figure 4: Division into four subproblems- III

$L = R$

We know that all the coins in $L$ and $R$ are genuine. If there is a fake coin, it must be in $E$. Let $L' = \{c_1, c_9\}$, $R' = \{c_{10}, c_{11}\}$ and $E' = \{c_{12}\}$. Compare $L'$ and $R'$.

If $L' = R'$, then the only possibilities are either all the coins are genuine or $c_{12}$ is a fake coin (that is, it is either lighter or heavier). By comparing $c_{12}$ with a genuine coin, we can determine whether it is genuine, lighter or heavier.

If $L' < R'$, we have three possibilities since one of $c_9$, $c_{10}$, and $c_{11}$ must be a fake coin.

- $c_9$ is lighter, and $c_{10}$, $c_{11}$, and $c_{12}$ are genuine

- $c_{10}$ is heavier, and $c_9$, $c_{11}$ and $c_{12}$ are genuine

- $c_{11}$ is heavier, $c_9$, $c_{10}$ and $c_{12}$ are genuine

We compare $c_{10}$ and $c_{11}$ to determine which one of the three alternatives holds. If $c_{10}$ and $c_{11}$ are equal in weight, then $c_9$ is lighter. Otherwise, which ever coin is heavier between them is the counterfeit and it is heavier.

If $L' > R'$, we once again have three possibilities since one of $c_9$, $c_{10}$, and $c_{11}$ must be fake. The possibilities are $c_9$ is heavier, $c_{10}$ is lighter, or $c_{11}$ is lighter. We compare $c_{10}$ and $c_{11}$ to resolve this ambiguity.

$L < R$

In this case, we know that the coins in $E$ are genuine and either one of the coins in $L$ is lighter or one of the coins in $R$ is heavier. Altogether we have 8 possibilities.

Let $L' = \{c_4, c_6, c_9\}$, $R' = \{c_3, c_7, c_8\}$, and $E' = \{c_1, c_2, c_5\}$. Compare $L'$ and $R'$.

If $L' = R'$, then we have three possibilities: $c_1$ is lighter, $c_2$ is lighter, or $c_5$ is heavier. By comparing $c_1$ and $c_2$, we can resolve the uncertainty.

If $L' < R'$, we again have three possibilities: $c_4$ is lighter, $c_7$ is heavier, or $c_8$ is heavier. By comparing $c_7$ and $c_8$, we can find the fake coin and its relative weight.

If $L' > R'$, we have only two possibilities: $c_6$ is heavier or $c_3$ is lighter. A comparison between $c_6$ and $c_7$ should tell us which is the case.

$L > R$

This case can be handled similar to the previous case.

In conclusion, we have shown that we need three comparisons to determine whether there is a fake coin, and if so which coin it is and whether it is lighter or heavier.

### Problem 6: A Stack of Fake Coins
There are 10 stacks of 10 identical-looking coins. All of the coins in one of these stacks are counterfeit, and all the coins in the other stacks are genuine. Every genuine coin weighs 10 grams, and every fake weighs 11 grams. You have an analytical scale that can determine the exact weight of any number of coins. What is the minimum number of weighings needed to identify the stack with the fake coins?

### Solution: A Stack of Fake Coins
One weighing is sufficient. Let $S_1, S_2, \ldots, S_{10}$ denote the ten stacks. Create a group of coins by selecting $i$ coins from stack $S_i$ for every $1 \leq i \leq 10$. Use the scale to determine their weight. If the weight is $w$, the counterfeit stack is then $S_{w-550}$.

There are $\frac{(10)(10+1)}{2} = 55$ coins in the group. Each genuine coin weighs 10 grams, so the entire group, would weigh 550 grams if every $S_i$ is a stack of genuine coins. However, there are between 1 and 10 fake coins in the group depending on which stack contains the fake coins. For any $1 \leq i \leq 12$, if $S_i$ is the stack of fake coins, the weight of the group would be $550 + i$. Hence, subtracting 550 from the weight of the group should give us the index of the stack of fake coins.

# 4   Problems

## Problem 7: Max-Min Weights

Given $n > 1$ items and a two-pan balance scale with no weights, determine the lightest and the heaviest items in $\lceil 3n/2 \rceil - 2$ weighings.

## Problem 8: MAX-SUM

Use the divide-and-conquer approach to write an efficient recursive algorithm that finds the maximum sum in any contiguous sublist of a given list of $n$ real (positive or negative) values. Analyse your algorithm, and show the results in order notation. Can you do better? Obtain a linear-time algorithm.

## Problem 9: Tromino Tilings

For each of the three cases, prove or disprove that for every $n > 0$ all the boards of the following can be tiled by right trominoes.

A *tiling* is a cover of the of board with no overlaps.

1. $3^n \times 3^n$

2. $5^n \times 5^n$

3. $6^n \times 6^n$

Recall that right trominoes are L-shaped tiles formed by three adjacent squares. In a tiling, trominoes can be oriented in different ways, but they should cover all the squares of the board exactly with no overlaps.

## Problem 10: Pancake Sorting

There are $n$ pancakes, all of different sizes, that are stacked on top of each other. You are allowed to slip a spatula under one of the pancakes and flip over the whole stack above the spatula. The objective is to arrange the pancakes according to their size with the biggest at the bottom. Design an algorithm for solving this puzzle and determine the number of flips made by the algorithm in the worst case.

## Problem 11: The Josephus Problem

We have $n$ people numbered (clockwise) 1 to $n$ around a circle. We eliminate every second person until only one survives. Determine the survivor's number.